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A SUZUKI-TYPE FIXED POINT THEOREM FOR NONLINEAR CONTRACTIONS

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Abstract. We introduce the notion of admissible functions and show that the family of L-functions introduced by Lim in [Nonlinear Anal. 46(2001), 113–120] and the family of test functions introduced by Geraghty in [Proc. Amer. Math. Soc., 40(1973), 604–608] are admissible. Then we prove that if ϕ is an admissible function, (X,d) is a complete metric space, and T is a mapping on X such that, for $\alpha(s)=\phi(s)/s$, the condition $\big(1+\alpha(d(x,Tx))\big)^{-1}d(x,Tx)<d(x,y)$ implies $d(Tx,Ty)<\phi(d(x,y))$, for all $x,y\in X$, then T has a unique fixed point. We also show that our fixed point theorem characterizes the metric completeness of X.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{Z}^+$ the set of nonnegative integers, and by $\mathbb{R}^+$ the set of nonnegative real numbers. Given a set X and a mapping $T:X\to X$, the nth iterate of T is denoted by T^n so that $T^2x=T(Tx)$, $T^3x=T(T^2x)$ and so on. A point $x\in X$ is called a *fixed point* of T if $Tx=x$.

Let (X,d) be a metric space. A mapping $T:X\to X$ is called a *contraction* if there is $r\in[0,1)$ such that $d(Tx,Ty)\le rd(x,y)$, for all $x,y\in X$. The following famous theorem is referred to as the Banach contraction principle.

Theorem 1.1 (Banach [2])**.** *If (X,d) is a complete metric space, then every contraction T on X has a unique fixed point.*

The Banach fixed point theorem is very simple and powerful. It became a classical tool in nonlinear analysis with many generalizations; see [3, 4, 5, 8, 13, 15, 16, 21, 22, 23, 24, 26, 27]. For instance, the following result due to Boyd and Wong [3] is a great generalization of Theorem 1.1.

Theorem 1.2 (Boyd and Wong [3])**.** *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume there exists a function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ which is upper semi-continuous from the right, $\phi(s)<s$ for $s>0$, and*

$$\forall\, x,y\in X,\quad d(Tx,Ty)\le\phi(d(x,y)). \tag{1.1}$$

Then T has a unique fixed point.

Another interesting generalization of Banach contraction principle was given by Meir and Keeler [15] as follows.

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Theorem 1.3 (Meir and Keeler [15]). *Let (X,d) be a complete metric space and let T be a Meir-Keeler contraction on X, i.e., for every $\varepsilon>0$, there exists $\delta>0$ such that*

$$\forall\, x,y\in X\ \ (\varepsilon\le d(x,y)<\varepsilon+\delta \implies d(Tx,Ty)<\varepsilon). \tag{1.2}$$

Then T has a unique fixed point.

Lim [14] introduced the notion of L-functions and proved a characterization of Meir-Keeler contractions that shows how much more general is Meir-Keeler's result than Boyd-Wong's. A function $\phi:[0,\infty)\to[0,\infty)$ is called an *L-function* if $\phi(0)=0$, $\phi(s)>0$ for $s>0$, and for every $s>0$ there exists $\delta>0$ such that $\phi(t)\le s$ for all $t\in[s,s+\delta]$.

Theorem 1.4 (Lim [14], see also [25]). *Let (X,d) be a metric space and let T be a mapping on X. Then T is a Meir-Keeler contraction if and only if there exists an L-function ϕ such that*

$$\forall\, x,y\in X,\quad d(Tx,Ty)<\phi(d(x,y)).$$

There is an example of an incomplete metric space X on which every contraction has a fixed point, [6]. This means that Theorem 1.1 cannot characterize the metric completeness of X. Recently, Suzuki in [26] proved the following remarkable generalization of the classical Banach contraction principle that characterizes the metric completeness of X.

Theorem 1.5 (Suzuki [26]). *Define a function $\theta:[0,1)\to(1/2,1]$ by*

$$\theta(r)=\begin{cases}1, & \text{if } 0\le r\le(\sqrt5-1)/2;\\ (1-r)r^{-2}, & \text{if } (\sqrt5-1)/2\le r\le 1/\sqrt2;\\ (1+r)^{-1}, & \text{if } 1/\sqrt2\le r<1.\end{cases} \tag{1.3}$$

Let (X,d) be a metric space. Then X is complete if and only if every mapping T on X satisfying the following has a fixed point:

- *There exists $r\in[0,1)$ such that*

$$\forall\, x,y\in X\ \big(\theta(r)d(x,Tx)\le d(x,y) \implies d(Tx,Ty)\le rd(x,y)\big). \tag{1.4}$$

The above Suzuki's generalized version of Theorem 1.1 initiated a lot of work in this direction and led to some important contribution in metric fixed point theory. Several authors obtained variations and refinements of Suzuki's result; see [9, 11, 12, 17, 19, 20].

A mapping T on a metric space X is called *contractive* if $d(Tx,Ty)<d(x,y)$, for all $x,y\in X$ with $x\neq y$. Edelstein in [7] proved that, on compact spaces, every contractive mapping possesses a unique fixed point theorem. Then in [27] Suzuki generalized Edelstein's result as follows.

Theorem 1.6 (Suzuki [27]). *Let (X,d) be a compact metric space and let T be a mapping on X. Assume that*

$$\forall\, x,y\in X\ \Big(\frac12 d(x,Tx)<d(x,y) \implies d(Tx,Ty)<d(x,y)\Big). \tag{1.5}$$

Then T has a unique fixed point.

It is interesting to note that, although the above Suzuki's theorem generalizes Edelstein's theorem in [7], these two theorems are not of the same type [27].

Recently, the author proved the following fixed point theorem for contractive mapping which is a Susuki-type generalization of [10, Theorem 1.1] and characterizes metric completeness.

Theorem 1.7 (Abtahi [1]). *A metric space (X, d) is complete if and only if every mapping $T : X \to X$ satisfying the following two conditions has a fixed point;*

(i) $(1/2)d(x, Tx) < d(x, y)$ *implies* $d(Tx, Ty) < d(x, y)$*, for all* $x, y \in X$.

(ii) *There exists a point* $x \in X$ *such that, for any two subsequences* $\{x_{p_n}\}$ *and* $\{x_{q_n}\}$ *of the iterations* $x_n = T^n x$, $n \in \mathbb{N}$*, if* $d(x_{p_n}, Tx_{p_n}) \le d(x_{p_n}, x_{q_n})$ *for all* n*, and* $\Delta_n \to 1$*, then* $\delta_n \to 0$*, where*

$$\delta_n = d(x_{p_n}, x_{q_n}), \quad \Delta_n = d(Tx_{p_n}, Tx_{q_n})/\delta_n.$$

Remark 1.8. In part (i) of the above theorem, $1/2$ is the best constant.

2. Existence of fixed points for nonlinear contractions

Definition 2.1. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. Given a metric space (X, d), a mapping $T : X \to X$ is called a *generalized ϕ-contraction* if

$$\forall\, x, y \in X \ \Big(x \neq y,\ d(x, Tx) \le d(x, y) \implies d(Tx, Ty) < \phi(d(x, y))\Big). \tag{2.1}$$

We call ϕ *admissible* if, for every metric space X, for every generalized ϕ-contraction T on X, and for every choice of initial point $x \in X$, the iterations $x_n = T^n x$, $n \in \mathbb{N}$, form a Cauchy sequence.

Theorem 2.2. *Every L-function is admissible.*

Proof. Let ϕ be an L-function and let T be a generalized ϕ-contraction on a metric space X. Fix $x \in X$ and let $x_n = T^n x$, $n \in \mathbb{N}$. If $d(x_m, x_{m+1}) = 0$, for some m, then $x_n = x_m$ for $n \ge m$ and there is nothing to prove. Assume that $d(x_n, x_{n+1}) > 0$ for all n. Since $d(x_n, Tx_n) \le d(x_n, Tx_n)$ and $x_n \ne x_{n+1}$, condition (2.1) implies that, for every $n \in \mathbb{N}$,

$$d(x_{n+1}, x_{n+2}) < \phi(d(x_n, x_{n+1})) \le d(x_n, x_{n+1}).$$

This shows that the sequence $\{d(x_n, x_{n+1})\}$ is strictly decreasing and thus it converges to some point $s \ge 0$. If $s > 0$, since ϕ is an L-function, there is $\delta > 0$ such that $\phi(t) \le s$ for $s \le t \le s+\delta$. Take $n \in \mathbb{N}$ large enough so that $s \le d(x_n, x_{n+1}) \le s+\delta$. Then

$$d(x_{n+1}, x_{n+2}) < \phi(d(x_n, x_{n+1})) \le s,$$

which is a contradiction. Hence $d(x_n, x_{n+1}) \to 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. To this end we adopt the same method used by Suzuki in [25]. Fix $\varepsilon > 0$ and let $s = \varepsilon/2$. Since ϕ is an L-function, there exists $\delta \in (0, s)$ such that $\phi(t) \le s$ for $s \le t \le s + \delta$. Since $d(x_n, x_{n+1}) \to 0$, there is $N \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \delta$ for $n \ge N$. We show that

$$d(x_n, x_{n+m}) < \delta + s \le \varepsilon, \qquad (n \ge N,\ m \in \mathbb{N}). \tag{2.2}$$

For every $n \ge N$, we prove (2.2) by induction on m. It is obvious that (2.2) holds for $m = 1$. Assume that (2.2) holds for some $m \in \mathbb{N}$. Then $\phi(d(x_n, x_{n+m})) \le$

s. Now, if $d(x_n, Tx_n) \leq d(x_n, x_{n+m})$ then (2.1) shows that $d(x_{n+1}, x_{n+m+1}) < \phi(d(x_n, x_{n+m}))$ and thus

$$d(x_n, , x_{n+m+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m+1}) < \delta + s \leq \varepsilon.$$

If $d(x_n, x_{n+m}) < d(x_n, Tx_n)$ then $d(x_n, x_{n+m}) < \delta$ and thus

$$d(x_n, , x_{n+m+1}) \leq d(x_n, x_{n+m}) + d(x_{n+m}, x_{n+m+1}) < \delta + \delta \leq \delta + s \leq \varepsilon.$$

Therefore (2.2) is verified and $\{x_n\}$ is a Cauchy sequence. $\square$

As in [10], we define $\mathbf{S}$ to be the class of all functions $\alpha : \mathbb{R}^+ \to [0, 1]$ such that, for any sequence $\{s_n\}$ of positive numbers, if $\alpha(s_n) \to 1$ then $s_n \to 0$.

Theorem 2.3. *If $\alpha \in \mathbf{S}$, the function $\phi(s) = \alpha(s)s$ is admissible.*

Proof. Let $\alpha \in \mathbf{S}$ and define $\phi(s) = \alpha(s)s$. Let T be a generalized ϕ-contraction on a metric space X, let $x \in X$ and let $x_n = T^n x$, $n \in \mathbb{N}$. Let $s_n = d(x_n, x_{n+1})$. As in the proof of Theorem 2.2, we assume that $s_n > 0$ for all n. Then $s_{n+1} < \alpha(s_n)s_n$ and thus $s_n \to s$ for some point $s \geq 0$. If $s > 0$ then $s_{n+1}/s_n \to 1$ and thus $\alpha(s_n) \to 1$. Since $\alpha \in \mathbf{S}$, we must have $s = 0$ which is a contradiction. Hence $s = 0$ and $d(x_n, x_{n+1}) \to 0$.

For every $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $d(x_m, x_{m+1}) < 1/n$ for $m \geq k_n$. If $\{x_n\}$ is not a Cauchy sequence, there exist $\varepsilon > 0$ and sequences $\{p_n\}$ and $\{q_n\}$ of positive integers such that $q_n > p_n \geq k_n$ and $d(x_{p_n}, x_{q_n}) \geq \varepsilon$. We also assume that q_n is the least such integer so that $d(x_{p_n}, x_{q_n-1}) < \varepsilon$. Therefore,

$$\varepsilon \leq d(x_{p_n}, x_{q_n}) \leq d(x_{p_n}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}) < \varepsilon + 1/n.$$

This shows that $s_n \to \varepsilon$. Since we have, for every $n \in \mathbb{N}$,

$$d(x_{p_n}, Tx_{p_n}) \leq d(x_{p_n}, x_{q_n}) < d(x_{p_n}, x_{q_n}),$$

condition (2.1) shows that $d(x_{p_n+1}, x_{q_n+1}) < \alpha(s_n)s_n$. Hence we have

$$\begin{aligned} s_n = d(x_{p_n}, x_{q_n}) &\leq d(x_{p_n}, x_{p_n+1}) + d(x_{p_n+1}, x_{q_n+1}) + d(x_{q_n+1}, x_{q_n}) \\ &< 2/n + \alpha(s_n)s_n. \end{aligned}$$

Dividing the above inequality by s_n, since $\alpha(s_n) \leq 1$, we get $\alpha(s_n) \to 1$ and thus $s_n \to 0$ which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. $\square$

Definition 2.4. A function $\alpha : \mathbb{R}^+ \to (0, 1]$ is said to be of class Ψ, written $\alpha \in \Psi$, if the function $\phi(s) = \alpha(s)s$ is admissible and, moreover, there exists $\delta > 0$ such that

$$0 < t < \delta,\ 0 < s < \alpha(t)t \implies \alpha(t) \leq \alpha(s). \tag{2.3}$$

Given two points x and y in a metric space (X, d), by $\alpha(x, y)$ we always mean $\alpha(d(x, y))$.

Example. Every decreasing function $\alpha : \mathbb{R}^+ \to (0, 1]$ is of class Ψ. For example, if $\alpha(s) = (1 + s)^{-1}$, then $\alpha \in \Psi$.

Theorem 2.5. *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that there is a function $\alpha \in \Psi$ such that*

$$\big(1 + \alpha(x, Tx)\big)^{-1} d(x, Tx) < d(x, y) \implies d(Tx, Ty) < \alpha(x, y)d(x, y), \tag{2.4}$$

holds for every $x, y \in X$. Then T has a unique fixed point.

Proof. First, let us prove the uniqueness part of the theorem. If $z \in X$ is a fixed point of T and $y \neq z$ then

$$\big(1+\alpha(z,Tz)\big)^{-1} d(z,Tz) < d(z,y),$$

and thus by (2.4) we have $d(Tz,Ty) < d(z,y)$. Since $Tz = z$, we must have $Ty \neq y$, i.e., y is not a fixed point of T.

Now, we prove the existence of the fixed point. Take two points $x, y \in X$ with $x \neq y$. If $d(x,Tx) \leq d(x,y)$ then $\big(1+\alpha(x,Tx)\big)^{-1} d(x,Tx) < d(x,y)$, because $\alpha(x,Tx) > 0$ and $d(x,y) > 0$. Hence T satisfies condition (2.1) with $\phi(s) = \alpha(s)s$. Fix $x \in X$ and define $x_n = T^n x$, $n \in \mathbb{N}$. Since the function $\phi(s) = \alpha(s)s$ is admissible, the sequence $\{x_n\}$ is Cauchy. Since X is complete, there is $z \in X$ such that $x_n \to z$. Next, we show that $Tz = z$.

If $x_m = Tx_m$ for some m, the $x_n = z$ for $n \geq m$ and $Tz = z$. We assume that $x_n \neq Tx_n$ for all n. Since $\alpha \in \Psi$, condition (2.3) holds for some $\delta > 0$. Take a positive number N such that $d(x_n, Tx_n) < \delta$ for $n \geq N$. Then

$$0 < d(Tx_n, T^2x_n) < \alpha(x_n, Tx_n) d(x_n, Tx_n), \qquad (n \geq N),$$

and condition (2.3) shows that $\alpha(x_n, Tx_n) \leq \alpha(Tx_n, T^2x_n)$, for $n \geq N$, so that

$$\frac{1}{1+\alpha(x_n,Tx_n)} + \frac{\alpha(x_n,Tx_n)}{1+\alpha(Tx_n,T^2x_n)} \leq 1. \tag{2.5}$$

We claim that

$$\forall\, n \geq N, \quad \begin{cases} \big(1+\alpha(x_n,Tx_n)\big)^{-1} d(x_n,Tx_n) < d(x_n,z), \\ \quad \text{or} \\ \big(1+\alpha(Tx_n,T^2x_n)\big)^{-1} d(Tx_n,T^2x_n) < d(x_{n+1},z). \end{cases} \tag{2.6}$$

If (2.6) fails to hold, then, for some $n \geq N$, we have

$$d(x_n,z) \leq \big(1+\alpha(x_n,Tx_n)\big)^{-1} d(x_n,Tx_n),$$
$$d(x_{n+1},z) \leq \big(1+\alpha(Tx_n,T^2x_n)\big)^{-1} d(Tx_n,T^2x_n).$$

Using (2.5), we have

$$\begin{aligned} d(x_n,Tx_n) &\leq d(x_n,z) + d(Tx_n,z) \\ &\leq \big(1+\alpha(x_n,Tx_n)\big)^{-1} d(x_n,Tx_n) + \big(1+\alpha(Tx_n,T^2x_n)\big)^{-1} d(Tx_n,T^2x_n) \\ &< \Big[\big(1+\alpha(x_n,Tx_n)\big)^{-1} + \big(1+\alpha(Tx_n,T^2x_n)\big)^{-1}\alpha(x_n,Tx_n)\Big] d(x_n,Tx_n) \\ &\leq d(x_n,Tx_n). \end{aligned}$$

This is absurd and thus (2.6) must hold. Now condition (2.4) together with (2.6) imply that

$$\forall n \geq N, \quad d(x_{n+1},Tz) < \phi(d(x_n,z)) \text{ or } d(x_{n+2},Tz) < \phi(d(x_{n+1},z)). \tag{2.7}$$

Since $x_n \to z$ and $\phi(s) \leq s$, condition (2.7) implies the existence of a subsequence of $\{x_n\}$ that converges to Tz. This shows that $Tz = z$. □

The following theorem states that, for a certain family of functions $\alpha \in \Psi$, the coefficient $1/(1+\alpha)$ in Theorem 2.5 is the best.

Theorem 2.6. *Let the function $\alpha \in \Psi$ satisfy the following condition;*

$$\alpha_0 = \liminf_{s \to 0+} \alpha(s) > 1/\sqrt{2}. \tag{2.8}$$

Then, for every constant η with $\eta > 1/(1+\alpha_0)$, there exist a complete metric space (X, d) and a mapping $T : X \to X$ such that T does not have a fixed point and

$$\forall\, x, y \in X,\ \big(\eta d(x, Tx) < d(x, y) \implies d(Tx, Ty) < \alpha(x, y) d(x, y)\big).$$

Proof. Take a number $r \in (1/\sqrt{2}, \alpha_0)$ such that $(1 + r)^{-1} < \eta$. The proof of Theorem 3 in [26] shows that there exist a closed and bounded subset X of $\mathbb{R}$ and a mapping $T : X \to X$ such that T does not have a fixed point and

$$\forall\, x, y \in X\ \Big((1 + r)^{-1}|x - Tx| < |x - y| \implies |Tx - Ty| \le r|x - y|\Big). \tag{2.9}$$

Since $r < \liminf_{s \to 0+} \alpha(s)$, there exists $\delta > 0$ such that $r < \alpha(s)$ for $s \in (0, \delta)$. Since X is bounded, there is a constant M such that $|x - y| < M\delta$, for all $x, y \in X$. Now, define a metric d on X by

$$d(x, y) = \frac{1}{M}|x - y|, \qquad (x, y \in X).$$

For $x, y \in X$, if $\eta d(x, Tx) < d(x, y)$ then $(1+r)^{-1} d(x, Tx) < d(x, y)$. Now, condition (2.9) and the fact that $d(x, y) < \delta$ shows that

$$d(Tx, Ty) \le r d(x, y) < \alpha(d(x, y))d(x, y).$$

□

Example. For the function $\alpha(s) = (1 + s)^{-1}$, we have $\alpha_0 = 1$. Hence α satisfies the condition in Theorem 2.6.

3. Metric Completion

In this section, we discuss the metric completeness. Let X be a nonempty set. We say that two metrics d and ρ on X are equivalent if they generate the same topology and the same Cauchy sequences. Given a metric ρ on X, we denote the family of all metrics d on X equivalent to ρ by $\mathcal{E}_\rho$. It is obvious that (X, ρ) is complete if and only if (X, d), for some $d \in \mathcal{E}_\rho$, is complete if and only if (X, d), for all $d \in \mathcal{E}_\rho$, is complete. For a function $\alpha \in \Psi$, we define

$$\alpha_0 = \liminf_{s \to 0+} \alpha(s),$$

and we denote by Ψ^+ the family of those functions $\alpha \in \Psi$ with $\alpha_0 > 0$.

Theorem 3.1. *For a metric space (X, ρ) the following are equivalent:*

(1) The space (X, ρ) is complete.
(2) For every $\alpha \in \Psi$ and $d \in \mathcal{E}_\rho$, every mapping T satisfying (2.4) has a fixed point.
(3) For some $\alpha \in \Psi^+$ and $\eta \in (0, 1/2]$, and for all $d \in \mathcal{E}_\rho$, every mapping T satisfying the following condition has a fixed point;

$$\forall\, x, y \in X,\ \big(\eta d(x, Tx) < d(x, y) \implies d(Tx, Ty) < \alpha(x, y)d(x, y)\big). \tag{3.1}$$

Proof. The implication (1) $\Rightarrow$ (2) follows from Theorem 2.5. The implication (2) $\Rightarrow$ (3) is clear because, for $\eta \le 1/2$, condition (3.1) implies condition (2.4).

To prove (3) $\Rightarrow$ (1), towards a contradiction, assume that the metric space (X, ρ) is not complete. Take a number $r \in (0, \alpha_0)$ and let δ be a positive number such that $r < \alpha(s)$ for all $s \in (0, \delta)$. Define a metric d on X as follows:

$$d(x, y) = \delta \frac{\rho(x, y)}{1 + \rho(x, y)}, \qquad (x, y \in X).$$

Then $d \in \mathcal{E}_\rho$ and thus (X, d) is not complete. The proof of Theorem 4 in [26] shows that there exists a mapping $T : X \to X$ with no fixed point such that

$$\forall\, x, y \in X,\ \big(\eta d(x, Tx) < d(x, y) \implies d(Tx, Ty) \le r d(x, y)\big).$$

Since $d(x, y) < \delta$, we have $rd(x, y) < \alpha(x, y)d(x, y)$ and thus T satisfies (3.1). This is a contradiction. $\square$

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